

How the Square Root Algorithm works

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One of the algorithms taught at the elementary level is deriving the square root of a number using the long division method. Like many algorithms taught in high school, the focus is on the 'how' of performing the steps rather than the 'why' these steps produce the square root. This article focuses on developing conceptual understanding of the Square Root Algorithm.

The difference between the factorization method and the division algorithm is that the former gives only the exact value of the square root of a whole number which is a perfect square whereas the latter may be used to find the square root of any positive number. Secondly, while factorization works for numbers with small factors, it becomes tedious when numbers have large prime factors.

We will find $\sqrt{576}$ by the division algorithm and in the process, we will raise some questions.

	24
2	<u>576</u>
	-4
44	<u>176</u>
	-176
	0

Figure 1

Step 1: Make pairs of digits from the right, in this case we get 5 and 76.

Step 2: Select the left most pair (5 here) and find the greatest number (2 here) whose square is equal to or less than it. Subtract this square from the selected pair. Write 2 on the left as well as above (as part of the result). This is the first digit which will appear in the quotient.

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Step 3: Write the second pair from the left (76 here) on the right side of subtracted result (1 here), to get a new dividend (176 here).

Step 4: Double the first number (the quotient 2) and write it down on the left of the new dividend.

Step 5: Now, by trial and error find the greatest digit $__$ such that $(4__ \times __) \leq 176$. Write this second new digit (4 here) on the right side of 2 in the quotient, now the quotient is 24.

Step 6: Subtract this product $(4__ \times __)$ from the dividend. Repeat steps 3 to 6 until there are no more pairs to be brought down.

Step 7: If the last difference is zero, the square root is the quotient (here the square root of 576 is 24). If not, append pairs of zeroes and find the square root to the required degree of accuracy. At the same time place a decimal point to the right of the quotient at this stage. Future entries will go to the right of the decimal point.

Questions raised here (and often not addressed)

- Why do we make such pairs, and why do we start making pairs from the right?
- How do we make pairs if the number has a decimal part?
- Why do we double the quotient each time and write it on the left as part of the divisor?
- Why do we append a new digit to the divisor and why do we append the same digit to the quotient and multiply the new divisor with the appended digit?

Theoretical part of Algorithm

The algorithm is based on estimating the largest square within the given number (with the help of place-value) and then refining that estimate by adding layers to the estimated square (again with the help of place-value) to get as close to the given number as possible.

There are three main important parts to understand the algorithm.

- i. Understanding why we make pairs
- ii. Understanding the idea of the formula $(a + b)^2 = a^2 + 2ab + b^2$
- iii. Understanding the iteration

i. Understanding why we make pairs

- Square of the smallest 2-digit number i.e. 10 is the smallest 3-digit number i.e. 100. So, any square number of 1-digit or 2-digit must be the square of a 1-digit number
- Similarly, square of the smallest 3-digit number i.e. 100 is the smallest 5-digit number i.e. 10000. So, any square number of 3-digit to 4-digit is the square of a 2-digit number
- Similarly, any square number of 5-digit to 6-digit is the square of a 3-digit number
- And any square number of 7-digit to 8-digit is the square of a 4-digit number and so on...

So, generally speaking, the square of the smallest n digit number (i.e. 10^{n-1}) is the smallest $(2n - 1)$ -digit number (i.e. 10^{2n-2}). Therefore, a square number of $(2n - 1)$ -digits or $2n$ digits must be the square of an n -digit number. So, if the digits of a number of $2n - 1$ -digits or $2n$ digits are paired, then the number of pairs provides the number of digits of the square root.

Now, if the pairing is done from the left then sometimes the units digit would be left alone (for odd number of digits e.g. for 729, 72 and 9) and otherwise it would be along with the tens digit (for even number of digits e.g. for 1296, 12 and 96). This is not consistent. But if the pairing is done from the right, then we get a consistent system since the units digit is always clubbed with the tens. Also, when we pair the digits, we are essentially splitting the number in n parts as follows: $729 = 700 + 29$, $1296 = 1200 + 96$, $21316 = 20000 + 1300 + 16$, $182329 = 180000 + 2300 + 29$, etc.

ii. **Application of the formula.** $(a + b)^2 = a^2 + 2ab + b^2$

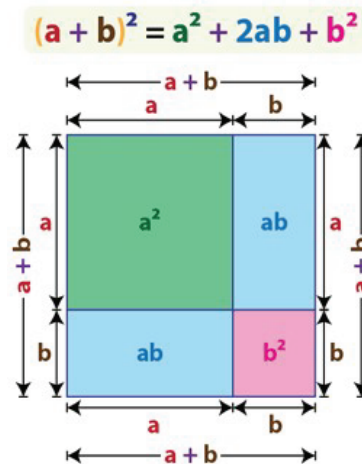


Figure 2

Any square number is of the form $(a + b)^2 = a^2 + 2ab + b^2$ which also can be written as $a^2 + (2a + b)b \dots (A)$

(We are interested in square numbers ≥ 100)

In order to find the square root $(a + b)$ from the square number $(a + b)^2$, we need to find an a and will subtract a^2 from $a^2 + (2a + b)b$ which will leave $(2a + b)b$. Then our work is to find such b for which $(2a + b)b$ holds.

Let us take an example to understand this part.

We have to find the square root of the number 4225.

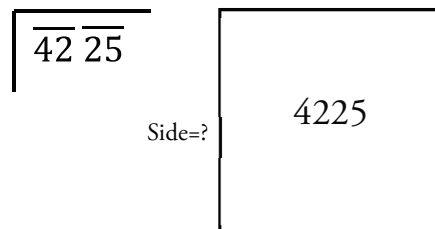


Figure 3

From 4225, we have to find a square number of the form $100x^2$ (since $100 < 4225 < 10000$)

The largest square of the form a^2 which is a multiple of 100 and less than 4225 is 3600

Therefore, remaining area = $4225 - 3600 = 625$

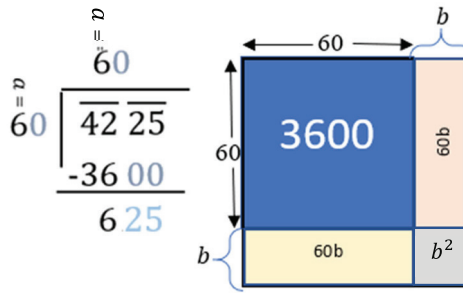


Figure 4

So, we need to find b such that

$$(2 \times 60b + b^2) = (120 + b)b = 625$$

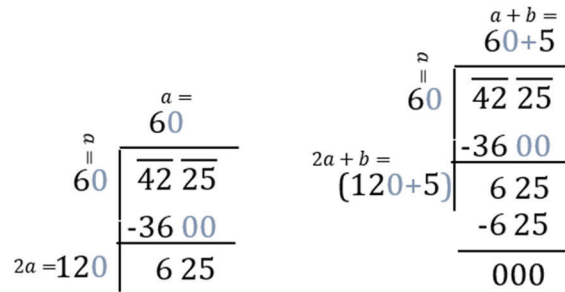


Figure 5

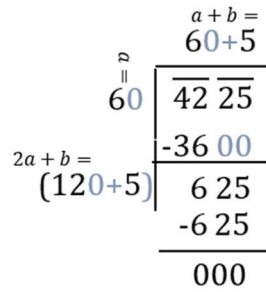


Figure 6

The process to find b involves some trial and error, and with some educated guess, especially if the given number is a perfect square.

We can check if $b = 5$ then $(120 + 5) \times 5 = 625$ (the last digits give us a clue to find b).

To find the square root of numbers with 5 or more digits, iteration is required where a' is the side of the next layer of the square i.e. $a + b$ and b' is the 2nd iteration.

iii. Understanding iteration. We will understand the iteration in two cases. The first case will be with the square of a 3-digit number and more. The later case will be with a number which is not a perfect square.

Case 1: Square of a 3-digit number. (The square will be between 10000 and 1000000.)

The 3 digit number may be written as $100x + 10y + z$ (where x, y and z are single digit numbers). The first objective is to find the first square of the form $(100x)^2$. So, we will find a number $a = 100x$, the square of which may be 10000, 40000, 90000, 160000, 250000, 360000, 490000, 640000 or 810000.

The next part (orange in Figure 8) depends on the selection of $b = 10y$.

b should be chosen so that the orange part does not exceed the remaining part (as in Figure 9).

This orange and gold part together represent $(2a + b)b = (2 \times 100x + 10y) 10y$

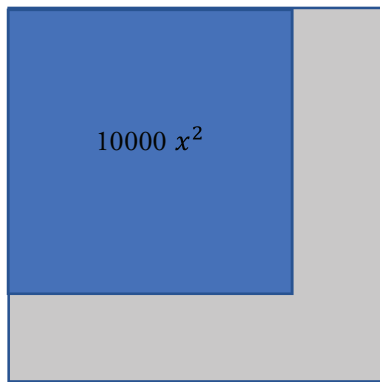


Figure 7

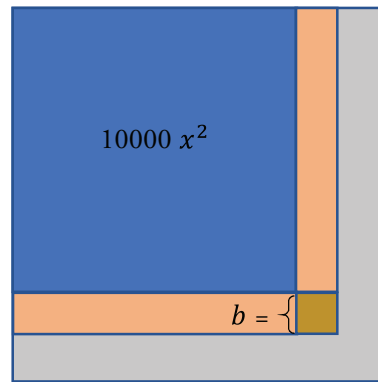


Figure 8

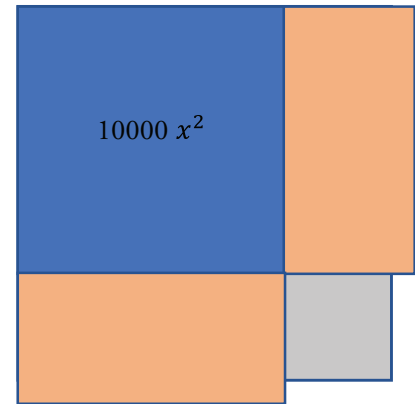


Figure 9

Finally, take $a' = a + b = 100x + 10y$ and then select $b' = z$ to cover the remaining gray part (as in Figure 8) which represents $(2a' + b') b' = \{2 \times (100x + 10y) + z\}z$

Now, $(100x)^2 + (2 \times 100x + 10y) 10y + \{2 \times (100x + 10y) + z\} z = (100x + 10y + z)^2$

Let us take an example (Figure 10) of finding the square root of 21316.

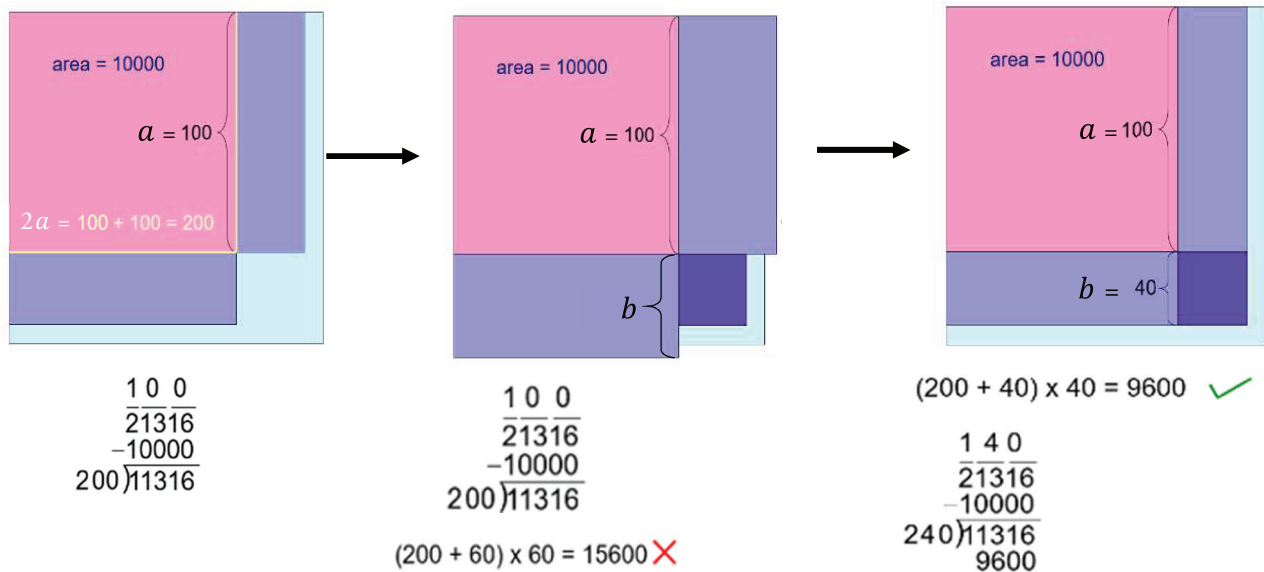


Figure 10

So, after choosing a suitable $b = 10y = 40$ in this example, we get $d' = a + b = 140$

Now we are to find b' .

Therefore, the square root is $a' + b' = (a + b) + b' = 146$ after two iterations (in Figures 10-11).

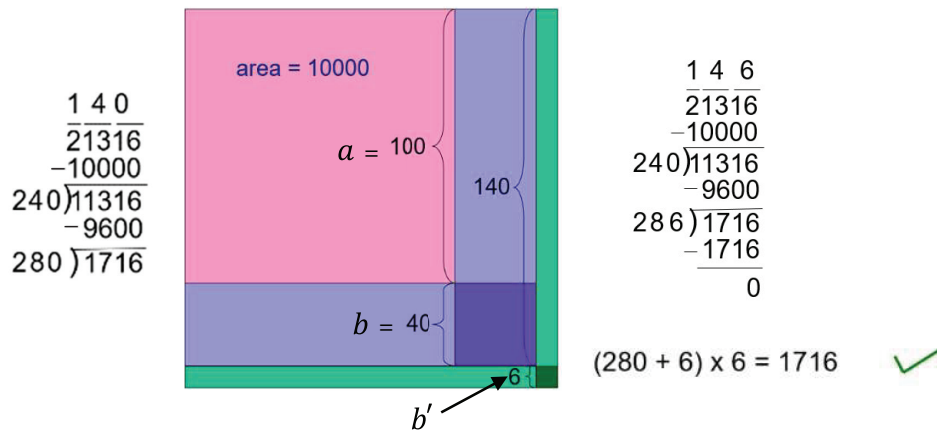
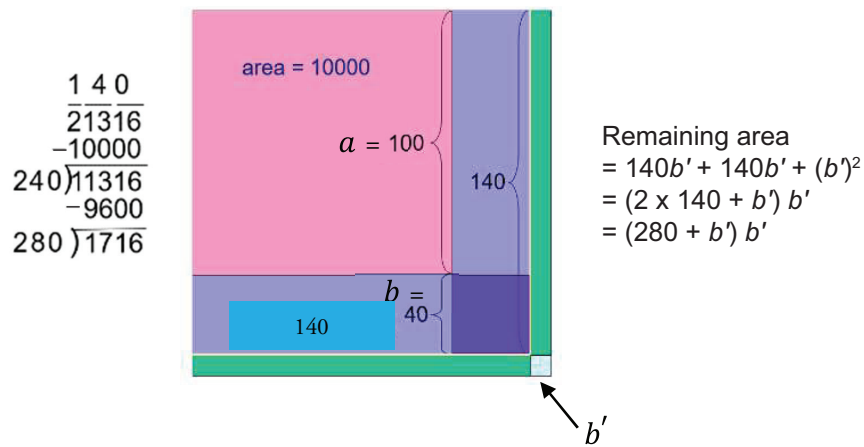
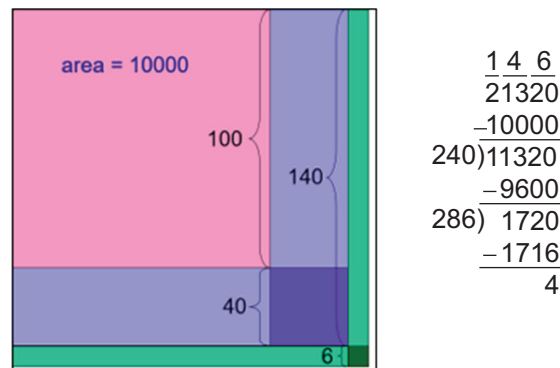


Figure 11

This process can be continued for squares of 4-digit numbers, squares of 5-digit numbers and so on.

Case 2: Now we will see its use to find the square root of a number which is not a perfect square. The iteration process is the same. Let us try to find the square root of 21320.

From the above case of finding square root of 21316, we can reach up to the step when 4 square area remains.



So, we will continue the iteration process by representing the number with decimal places like 21320.00 00 00...

$$\begin{array}{r}
 \overline{1\ 4\ 6\ 0\ 1\ 3} \\
 21320.000000 \\
 -10000 \\
 \hline
 24\)\ 11320 \\
 \underline{-9600} \\
 286\)\ 1720 \\
 \underline{-1716} \\
 2920\)\ 400 \\
 \underline{-0} \\
 29201\)\ 40000 \\
 \underline{-29201} \\
 292023\)\ 1079900 \\
 \underline{-876069}
 \end{array}$$

Here is the square root of 2

$$\begin{array}{r}
 \overline{1.41421} \\
 1\ 2.000000000000 \\
 \underline{1} \\
 24\ 100 \\
 \underline{96} \\
 281\ 400 \\
 \underline{281} \\
 2824\ 11900 \\
 \underline{11296} \\
 28282\ 60400 \\
 \underline{56564} \\
 282841\ 383600 \\
 \underline{282841} \\
 2828423\ 10075900
 \end{array}$$

$$\Rightarrow \sqrt{2} = 1.41421.....$$

For decimal numbers, the same process of pairing digits will be followed in the same direction from the whole number part to the decimal part of the number.

It is interesting to observe that the key theory behind this algorithm, i.e., $(a + b)^2$ is integral to upper primary algebra. However, even when its application, i.e., the division algorithm of finding the square root is included in the syllabus or textbooks, the connection is rarely made. This is an attempt to fill that gap.

References

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